Chapter 5

Circuit Theorems
Figure 5.1-1

Design problem involving a strain gauge bridge.

\[ v_l = v_i \times \frac{R + \Delta R}{2R} \]

\[ v_R = v_i \times \frac{R - \Delta R}{2R} \]

\[ v_0 = v_l - v_r \]

\[ = v_i \times \left( \frac{R + \Delta R}{2R} - \frac{R - \Delta R}{2R} \right) \]

\[ = v_i \times \frac{\Delta R}{R} \]
new concepts from ch. 5

- electric power for cities
- source transformations
- superposition principle
- Thévenin’s theorem
- Norton’s theorem
- maximum power transfer
electric power to the cities

generation $\rightarrow$ transmission $\rightarrow$ distribution

the network of electric power
Basic Components of Electric Power:

How electricity gets to you

When electricity leaves a power plant (1), its voltage is increased at a “step-up” substation (2). Next, the energy travels along a transmission line to the area where the power is needed (3). Once there, the voltage is decreased, or “stepped-down,” at another substation (4), and a distribution power line (5) carries the electricity until it reaches a home or business (6).

– EEI, Getting Electricity Where It’s Needed, May 2000
Electric Power Delivery Efficiency

Source: PJM Website
Section 5.3

source transformations

- procedure for transforming one source into another while retaining the terminal characteristics of the original source
- producing an equivalent circuit
- identical effect at terminals, not \textit{within} the circuits themselves
why transform?

- it may be easier to solve a circuit when the sources are all the same type (i.e., current or voltage)
Figure 5.3-1

Two equivalent circuits.
let’s transform this circuit...
to this circuit...
Figure 5.3-2

(a) Voltage source with an external resistor \( R \). (b) Current source with an external resistance \( R \).
for any applied load $R$

- both circuits must have the same characteristics
- let’s apply the extreme values of $R$
  - $R = 0$
  - $R = \infty$
When \( R = 0 \)

- we essentially have a short circuit
- therefore the short circuit current of each circuit must be equal

- for first circuit:
  - \( i = \frac{v_s}{R_s} \)

- for second circuit: \( i = i_s \), so...
  - \( i_s = \frac{v_s}{R_s} \)
When \( R = \infty \)

- we essentially have an open circuit
- therefore the open circuit voltage of each circuit must be equal
- for second circuit:
  - \( v = i_s R_p \)
- from the first circuit: \( v = v_s \), so...
  - \( v_s = i_s R_p \)
combining what we know...

- when $R = 0$
  - $i_s = v_s/R_s$

- when $R = \infty$
  - $v_s = i_s R_p$

- so from $R = 0$ to $\infty$

  $v_s = (v_s/R_s) R_p$

  Therefore $R_s = R_p$
dual circuits

circuits are said to be duals when the characterizing equations of one network can be obtained by simple interchange of \( v \) and \( i \) and \( G \) and \( R \)

\[ R_p = \frac{1}{G_p} \]

\[ i_s = v_s G_p \quad \text{and} \quad v_s = i_s R_s \]
examples: this circuit is equivalent to...

\[ R_s = 12 \, \Omega \]

36V

O → a → Rs → b → O
this one.....

\[ R_p = R_s = 12 \, \Omega \]

\[ v_s = i_s R_s \text{ or } i_s = \frac{v_s}{R_p} \]

\[ = \frac{36}{12} = 3 \, \text{A} \]

\[ i_s = ? \, \text{A} \]

\[ 3 \, \text{A} \]

\[ R_p = ? \, \Omega \]

\[ 12 \, \Omega \]
examples: make these circuits equivalent...

\[ R_s = 10 \, \Omega \]
how…..

$R_p = R_s = 10 \, \Omega$

$So \ v_s = 12V = i_s \cdot R_s$ or $i_s = \frac{v_s}{R_p}$

$= \frac{12}{10} = 1.2V$
examples: make these circuits equivalent…

```
12V
```

```
R_s
```

```
O   a
```

```
O   b
```

22
$R_p = R_s = 10 \, \Omega$

how.....

So $v_s = -12V = i_s \, R_s$ or $i_s = v_s / R_p =\frac{-12}{10} = -1.2A$
examples: make these circuits equivalent...

$\begin{align*}
    i_s &= 3 \text{ A} \\
    R_p &= 8 \Omega
\end{align*}$
how.....

\[ R_s = ? \Omega \]

\[ R_p = R_s = 8 \, \Omega \]

\[ V_s = ? V \]

So \( v_s = i_s R_s \) or \(-24 \, V\)
Figure 5.3-3
Method of source transformations.

(a) Method

Method

Set $i_s = \frac{v_s}{R_s}$
Set $R_p = R_s$

(b) Method

Set $v_s = i_s R_p$
Set $R_s = R_p$
Figure 5.3-5

The circuit of Example 5.3-2.
Resistances in ohms.

\[ i_s = \frac{V_s}{R_s} = \frac{3}{30} = 0.1A \]

\[ R_p = R_s \]
Figure 5.3-6
Source transformation steps for Example 5.3-2. All resistances in ohms.

\[ v_s = i_s R_p = 0.1 \times 12 = 1.2 \text{V} \]

\[ \begin{align*}
R_s &= R_p 
\end{align*} \]
Section 5.4

superposition principle (SP)

In a single element:

if the application of

\( i_1 \) yields \( v_1 \) and \( i_2 \) yields \( v_2 \) then:

\( i_1 + i_2 \) will yield \( v_1 + v_2 \)

the total effect of several causes acting simultaneously is equal to the sum of the effects of the individual causes acting one at a time
SP can help…

- how to apply SP to simplify analysis
- disable *all but one* source
- find *partial* response to that source
- disable all but the next source
- find *partial* response to that source
- iterate
- sum all the partial effects to get total
How to… continued

- set current sources to 0 (open circuits)
- solve for partial effect
- set voltage sources to 0 (short circuits)
- solve for partial effect
- sum effects
examples

\[ i_1 = \frac{6}{3+6} = \frac{6}{9} \]

\[ i_2 = \frac{2 \times 3}{3+6} = \frac{6}{9} \]

\[ i_m = i_1 + i_2 = \frac{6}{9} + \frac{6}{9} = \frac{12}{9} \]
examples

\[
i = i_1 + i_2 = 3 - \frac{7}{4} = \frac{5}{4} \text{ A}
\]

\text{KVL:}
\[-24 + 3i_1 + 2i_1 + 3i_1 = 0 \Rightarrow i_1 = 3 \text{ A}\]

\text{KCL:}
\[-i_2 - 7 + \frac{-3i_2 - 2i_2}{2} = 0 \Rightarrow i_2 = -\frac{7}{4}\]
Section 5.5

Thévenin’s theorem

- **GOAL:** reduce some complex part of a circuit to an equivalent source and a single element (for analysis)
- **THEOREM:** for any circuit of resistance elements and energy sources with a identified terminal pair, the circuit is replaceable by a series combination of \( v_t \) and \( R_t \)
Thévenin method

- If circuit contains resistors and ind. sources
  - Connect open circuit between a and b. Find $v_{oc}$
  - Deactivate source(s), calc. $R_t$ by circuit reduction

- If circuit has resistors and ind. & dep. sources
  - Connect open circuit between a and b. Find $v_{oc}$
  - Connect short circuit across a and b. Find $i_{sc}$
  - Connect 1-A current source from b to a. Find $v_{ab}$
    - NOTE: $R_t = v_{ab} / 1$ or $R_t = v_{oc} / i_{sc}$

- If circuit has resistors and only dep. sources
  - Note that $v_{oc} = 0$
  - Connect 1-A current source from b to a. Find $v_{ab}$
    - NOTE: $R_t = v_{ab} / 1$
Example: R & indep. source

\[ v_{oc} = 50 \times \frac{20}{20 + 5} = 40V \]

\[ R_t = 4 + 5 \parallel 20 = 8\Omega \]
Example: V & I source

\[
\frac{v_{oc}}{10} - \frac{-10}{10} + \frac{v_{oc}}{40} + 2 = 0
\]

\[v_{oc} = -8V\]

\[R_t = 4 + \frac{10}{40} = 12 \Omega\]
Example: indep. & dep. source

\[
-20 + 6i_1 - 2i + 6(i_1 - i_2) = 0 \\
6(i_2 - i_1) + 10 \times i_2 = 0 \\
i = i_1 - i_2 \\
\Rightarrow i_{sc} = i_2 = \frac{120}{136}
\]

\[
KVL: -20 + 6i - 2i + 6i = 0 \\
i = 2A \Rightarrow v_{oc} = 6i = 12V
\]

\[
R_t = \frac{v_{oc}}{i_{sc}} = 12 / \left(\frac{120}{136}\right) = 13.6\Omega
\]
Example: dep. source

\[ V_{oc} = 0 \]
\[ i_{sc} = 0 \Rightarrow R_t = \frac{V_{oc}}{i_{sc}} = \frac{0}{0} \]

\[ KCL: \frac{V_{ab} - 2i}{5} + \frac{V_{ab}}{10} - 1 = 0 \]
\[ v_{ab} = 10i \Rightarrow v_{ab} = \frac{50}{13} V \]

\[ R_t = \frac{V_{ab}}{1} \]
\[ R_t = \frac{V_{ab}}{1} = \frac{50}{13} \Omega \]

\[ R_t = \frac{50}{13} \Omega \]
Section 5.6

Norton’s theorem

- **GOAL:** reduce some complex part of a circuit to an equivalent source and a single element (for analysis)

- **THEOREM:** for any circuit of resistive elements and energy sources with a terminal pair, the circuit is replaceable by a parallel combination of $i_{sc}$ and $R_n$ (this is a source transformation of the Thevenin)
Norton equivalent circuit

\[ R_n = R_t \]
Norton method

—if circuit contains resistors and ind. sources
- Connect short circuit between a and b. Find $i_{sc}$
- Deactivate ind. source(s), calc. $R_n = R_t$ by circuit reduction

—if circuit has resistors and ind. & dep. sources
- Connect short circuit across a and b. Find $i_{sc}$
- Connect open circuit between a and b. Find $v_{oc} = v_{ab}$
- Connect 1-A current source from b to a. Find $v_{ab}$
  - NOTE: $R_n = R_t = \frac{v_{ab}}{1}$ or $R_n = R_t = \frac{v_{oc}}{i_{sc}}$

—if circuit has resistors and only dep. sources
- Note that $i_{sc} = 0$
- Connect 1-A current source from b to a. Find $v_{ab}$
  - NOTE: $R_n = R_t = \frac{v_{ab}}{1}$
Example: R & indep. source

\[ R_n = (8 + 4) \parallel 6 = 4k\Omega \]

\[ i_{sc} = \frac{15}{8 + 4} = 1.25mA \]
Example: v & c source

\[ R_n = 4 // 12 = 3\Omega \]

\[ \text{KCL}: -\frac{24}{4} - 3 + i_{sc} = 0 \]

\( \Rightarrow i_{sc} = 9A \)
Example: dep. & indep. source

\[ -5 + 500i + v_{ab} = 0 \]
\[ v_{ab} = -25(10i) \Rightarrow i = \frac{-v_{ab}}{250} \]
\[ v_{ab} = -5V \]
\[ R_t = \frac{v_{ab}}{i_{sc}} = \frac{-5}{-0.1} = 50\Omega \]

\[ v_{ab} = 0 \Rightarrow i = \frac{5}{500} = 10mA \]
\[ i_{sc} = -10i = -100mA \]
maximum power transfer

- what is it?
- often it is desired to gain maximum power transfer for an energy source to a load
  - examples include:
    - electric utility grid
    - signal transmission (FM radio receiver)
    - source \( \rightarrow \) load
maximum power transfer

how do we achieve it?

\[ v_t \text{ or } v_{sc} \]
maximum power transfer

\[ p = i^2 R_L \]

\[ i = \frac{v_s}{R_L + R_t} \]

\[ \therefore p = \left(\frac{v_s}{R_L + R_t}\right)^2 R_L \]

\[ \therefore \frac{dp}{dR_L} = v_s^2 \frac{(R_t + R_L)^2 - 2(R_t + R_L)R_L}{(R_t + R_L)^4} = 0 \]

\[ R_L = R_t \]
maximum power transfer theorem

So…

maximum power delivered by a source represented by its Thevenin equivalent circuit is attained when the load $R_L$ is equal to the Thevenin resistance $R_t$
Figure 5.7-3
Power actually attained as $R_L$ varies in relation to $R_t$.

$$P_{\text{max}} = \frac{v_s^2 R_L}{(2R_L)^2} = \frac{v_s^2}{4R_L}$$
efficiency of power transfer

how do we calculate it for a circuit?

\[ \eta = \frac{p_{out}}{p_{in}} \]

\[ p_{in} = v_s i = v_s \left( \frac{v_s}{R_L + R_t} \right) = \frac{v_s^2}{2R_L} \]

\[ p_{out} = p_{max} \left( \frac{v_s}{R_L + R_L} \right)^2 = \frac{v_s^2}{4R_L} \]

\[ \therefore \eta = \frac{p_{out}}{p_{in}} = 50\% \text{ max} \]
Norton equivalent circuits

\[ p = i^2 R \] in a Norton equivalent circuit we find that it, too, has a maximum when the load \( R_L \) is equal to the Norton resistance \( R_n = R_t \).
Example

\[ KVL: -6 + 10i - 2v_{ab} = 0 \]
\[ 10i - 8i = 6 \Rightarrow i = 3A \]
\[ v_{oc} = 4i = 12V \]

\[ R_L = R_t = 12\Omega \]

\[ P_{max} = \frac{v_{oc}^2}{4R_L} \]
\[ = \frac{12^2}{4 \times 12} = 3W \]